BETTI NUMBERS OF PARABOLIC $U(2,1)$-HIGGS BUNDLES MODULI SPACES.

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ABSTRACT. Let X be a compact Riemann surface together with a finite set of marked points. We use Morse theoretic techniques to compute the Betti numbers of the parabolic $U(2,1)$-Higgs bundles moduli spaces over X. We give examples for one marked point showing that the Poincaré polynomials depend on the system of weights of the parabolic bundle.

1. INTRODUCTION

The moduli spaces of stable parabolic Higgs bundles have been studied in [BY, T, Y] and have a rich structure, partially due to its relation with the representation space of the fundamental group of a punctured Riemann surface. This relationship was established by Simpson in [S].

The topology of the moduli $\mathcal{U}$ of stable $U(p,q)$-parabolic Higgs bundles with fixed generic weights and degrees has already been considered in [GLM]. This moduli space is a submanifold of the moduli space $\mathcal{M}$ of stable parabolic Higgs bundles of fixed degree, which was analised in the rank 2 case by Boden and Yokogawa in [BY] and in the rank 3 case by García-Prada, Muñoz and Gothen in [GGM]. In these two papers, the authors obtained the Betti numbers of the moduli spaces $\mathcal{M}$. Our purpose here is to calculate Betti numbers of $\mathcal{U}$ when $p+q=3$. Note that in this case, $\mathcal{U}$ is a submanifold of the moduli $\mathcal{M}$ studied in [GGM]. It is known that for fixed rank, the moduli spaces $\mathcal{M}$ of stable parabolic Higgs bundles, corresponding to different choices of degrees and generic weights, are diffeomorphic [GGM, T]. Our computations for $p+q=3$ produce counterexamples to this type of phenomena for the submanifolds $\mathcal{U}$, that is, they provide an example of the dependence of these moduli spaces on the generic weights of the parabolic structure.

We will use Morse theory one step forward than in [GLM] thanks to fixing the rank equal to 3. Higher ranks need to develop another tool called parabolic chains and will be done in the future. We start in Section 2, explaining the necessary definitions and results for the Morse theory involved and defining also the Morse function that we are going to use. In Section 3 we study certain critical subvarieties of this Morse function before and in Section 5 we introduce parabolic triples for another type of critical subvarieties.

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Sections 4 and 6 give explicit computations for the case of one puncture and Section 7 summarizes the results and give some low genus examples.

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2. Definitions and Morse theory

Let X be a compact Riemann surface of genus $g \geq 0$ together with a finite set of marked distinct points $x_1, \dots, x_s$. We denote $D = x_1 + \cdots + x_s$ the divisor on X defined by the punctures.

A parabolic bundle E over X consists of a holomorphic bundle with a parabolic structure, that is, weighted flags, one for each puncture in X,

$$\begin{array}{rcl} E_x & = & E_{x,1} \supset \cdots \supset E_{x,r(x)} \supset 0, \\ 0 & \leq & \alpha_1(x) < \cdots < \alpha_{r(x)} < 1. \end{array}$$

The set of all weights for all $x \in D$, $\alpha = \{\alpha_i(x); i = 1, \dots, r(x)\}$, is called *parabolic system of weights* of E.

A holomorphic map $f : E \to E'$ between parabolic bundles is called *parabolic* if $\alpha_i(x) > \alpha'_j(x)$ implies $f(E_{x,i}) \subset E'_{x,j+1}$ for all $x \in D$, and f *strongly parabolic* if $\alpha_i(x) \geq \alpha'_j(x)$ implies $f(E_{x,i}) \subset E'_{x,j+1}$ for all $x \in D$, where we denote by $\alpha'_j(x)$ the weights on E'. Also $\mathrm{ParHom}\,(E, E')$ and $\mathrm{SParHom}\,(E, E')$ will denote respectively the bundles of parabolic and strongly parabolic morphisms from E to E'. Finally, a *parabolic subbundle* of a parabolic bundle is a subbundle which inherits its parabolic structure from the parabolic bundle.

We write $m_{\alpha_i}(x) = \dim(E_{x,i}/E_{x,i+1})$ for the multiplicity of the weight $\alpha_i(x)$ at x. The parabolic degree and parabolic slope of E are defined as

$$\mathrm{pardeg}\,(E) = \deg(E) + \sum_{x \in D} \sum_{i=1}^{r(x)} m_{\alpha_i}(x)\alpha_i(x),$$

$$\mathrm{par}\mu\,(E) = \frac{\mathrm{pardeg}\,(E)}{\mathrm{rk}\,(E)}.$$

A parabolic bundle is called (semi)-stable if for every parabolic subbundle F of E, the parabolic slope satisfies $\mathrm{par}\mu\,(F) \leq \mathrm{par}\mu\,(E)$ (resp. $\mathrm{par}\mu\,(F) < \mathrm{par}\mu\,(E)$).

For parabolic bundles E there is a well-defined notion of parabolic dual E^*. It consists of the bundle $\mathrm{Hom}\,(E, \mathcal{O}(-D))$ and at each $x \in D$ a weighted filtration

$$E^*_x = E^*_{x,1} \supset \cdots E^*_{x,r(x)} \supset 0,$$
$$0 < 1 - \alpha_{r(x)}(x) < \cdots < 1 - \alpha_1 < 1.$$

In the case $\alpha_1 = 0$ we choose the following weights for the filtration,

$$0 \leq \alpha_1 < 1 - \alpha_{r(x)}(x) < \cdots < 1 - \alpha_2 < 1.$$

With this definition $E^{**} = E$ and $\mathrm{pardeg}\,(E^*) = -\,\mathrm{pardeg}\,(E)$.

A $GL(n,\mathbb{C})$-*parabolic Higgs bundle* is a pair (E,Φ) consisting of a parabolic bundle E and $\Phi \in H^0(\mathrm{SParEnd}\,(E)\otimes K(D))$, i.e. Φ is a meromorphic endomorphism valued one-form with simple poles along D whose residue at $p \in D$ is nilpotent with resect to the flag. A parabolic Higgs bundle is called (semi)-stable if for every Φ-invariant subbundle F of E, its parabolic slope satisfies $\mathrm{par}\mu\,(F) \leq \mathrm{par}\mu\,(E)$ (resp. $\mathrm{par}\mu\,(F) < \mathrm{par}\mu\,(E)$). We shall say that the weights are *generic* when every semistable Higgs bundle is stable, that is, there are no properly semistable parabolic Higgs bundles.

A $U(p,q)$-*parabolic Higgs bundle* on X is a parabolic Higgs bundle (E,Φ) such that $E = V \oplus W$, where V and W are parabolic vector bundles of rank p and q respectively, and

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : (V \oplus W) \to (V \oplus W) \otimes K(D),$$

where the non-zero components $\beta : W \to V \otimes K(D)$ and $\gamma : V \to W \otimes K(D)$ are strongly parabolic morphisms. Hence a $U(p,q)$-parabolic Higgs bundle is (semi)-stable if the slope (semi)-stability condition is satisfied for all Φ-invariant subbundles of the form $F = V' \oplus W'$, i.e. for all subbundles $V' \subset V$ and $W' \subset W$ such that

$$\beta : \quad W' \to V' \otimes K(D) \tag{1}$$

$$\gamma : \quad V' \to W' \otimes K(D). \tag{2}$$

Let us fix generic weights and topological invariants $\mathrm{rk}\,(E)$ and $\deg(E)$. The moduli space $\mathcal{M}_{GL(n,\mathbb{C})}$ of stable $GL(n,\mathbb{C})$-parabolic Higgs bundles was constructed using Geometric Invariant Theory by Yokogawa [Y], who also showed that it is a smooth irreducible complex variety.

By definition there is an injection from the moduli $\mathcal{U}_{(p,q)}$ of stable $U(p,q)$-parabolic Higgs bundles to the moduli $\mathcal{M}_{GL(p+q,\mathbb{C})}$ of stable $GL(p+q,\mathbb{C})$-parabolic Higgs bundles. Moreover, such an injection is an embedding, as shown in [GLM], so $\mathcal{U}_{(p,q)}$ is in fact a submanifold of $\mathcal{M}_{GL(p+q,\mathbb{C})}$. When it does not induce confusion, we will denote $\mathcal{U}_{(2,1)}$ and $\mathcal{M}_{GL(3,\mathbb{C})}$ by $\mathcal{U}$ and $\mathcal{M}$.

The Toledo invariant for the moduli of $U(p,q)$ parabolic Higgs bundles is studied in [GLM] and defined as $\tau = 2(q\,\mathrm{pardeg}\,(V) - p\,\mathrm{pardeg}\,(W))/(p+q)$. Thus, given $(E,\Phi) \in \mathcal{U}$ we have

$$\tau = \frac{2}{3}(\Delta - 3b + \sum_{x \in D} \alpha_1(x) + \alpha_2(x) - 2\eta(x)), \tag{3}$$

where we denote $a = \deg(V)$, $b = \deg(W)$, $\alpha_1(x)$ and $\alpha_2(x)$ the parabolic weights on V and $\eta(x)$ the parabolic weights on W over the punctures $x \in D$, and $\Delta = a + b$. We will use this notation in the following.

Proposition 1. *The map $V \oplus W \to (V \oplus W) \otimes L$, where L is a parabolic line bundle, induces an isomorphism from the moduli space $\mathcal{U}_{(p,q)}(a,b)$ of parabolic $U(p,q)$-Higgs bundles with fixed degrees (a,b) to the moduli space $\mathcal{U}_{(p,q)}(a',b')$ of parabolic $U(p,q)$-Higgs bundles with fixed degrees (a',b'), where $a' = a + pl$ and $b' = b + ql$.*

The map $V \oplus W \to V^* \oplus W^*$ induces an isomorphism of moduli spaces, from $\mathcal{U}_{(p,q)}(a,b)$ to $\mathcal{U}'_{(p,q)}(a',b')$, where $a' = -a$ and $b' = -b$. □

Let us to assume that $\Delta = a' + b' \equiv 0(3)$ and that the parabolic Toledo invariant τ satisfies $\tau \geq 0$.

The moduli space $\mathcal{U}$ of stable $U(p,q)$-parabolic Higgs bundles has been studied in [GLM], where the number of connected components is calculated using Bott-Morse theory. Here we shall fix later $p = 2$ and $q = 1$ to go one step further and give topological information about this moduli space.

Consider the action of $\mathbb{C}^*$ on $\mathcal{U}$ given in [GLM] as

$$\begin{aligned} \psi : \mathbb{C}^* \times \mathcal{U} &\rightarrow \mathcal{U} && (4)\\ (\lambda, (E, \Phi)) &\mapsto (E, \lambda\Phi). && (5) \end{aligned}$$

This restricts to a Hamiltonian action of $S^1 \subset \mathbb{C}^*$ on $\mathcal{U}$ and the moment map associated to this Hamiltonian action is defined by

$$f([E, \Phi]) = \|\Phi\|^2 = \frac{1}{\pi}\|\beta\|^2 + \frac{1}{\pi}\|\gamma\|^2, \tag{6}$$

where we are using a suitable Sobolev metric for the norm given by the Hermite-Einstein equations for the parabolic Higgs bundle (E, Φ) (see [S]).

Observe that $f : \mathcal{U} \rightarrow \mathbb{R}$ is the restriction of the moment map $f : \mathcal{M} \rightarrow \mathbb{R}$ used in [GGM]. That map was proper. Hence, f is also proper since $\mathcal{U}$ is a closed submanifold of $\mathcal{M}$. This fact together with a result of Frankel [F], proving that a proper moment map for a Hamiltonian circle action on a Kähler manifold is a perfect Bott-Morse function, give us that f is a perfect Bott-Morse function.

Hence, we have the following formula for the Poincaré polynomial of the manifold $\mathcal{U}$,

$$P_t(\mathcal{U}) = \sum_{\mathcal{N}} t^{\lambda_{\mathcal{N}}} P_t(\mathcal{N}), \tag{7}$$

where the sum runs over all critical submanifolds $\mathcal{N}$ of $\mathcal{U}$ for f and $\lambda_{\mathcal{N}}$ is the Morse index of f on $\mathcal{N}$.

The critical points of f are exactly the fixed points of the circle action. Moreover, the Morse index of f at a critical point equals the dimension of the negative weight space of the circle action on the tangent space [F].

Simpson's theorem gives us a criterion for (E, Φ) to be a critical point for the Morse function.

Theorem 2 ([S], Thm.8)**.** *The equivalence class of a stable parabolic Higgs bundle (E, Φ) is fixed under the action of S^1 if and only if it is a* parabolic complex variation of Hodge structure. *This means that E has a direct sum decomposition*

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_m$$

as parabolic bundles, such that Φ is strongly parabolic and of degree one with respect to this decomposition, in other words, the restriction $\Phi_l = \Phi|_{E_l} \in H^0(\mathrm{SParHom}\,(E_l, E_{l+1}) \otimes K(D))$. Also $\Phi_l \neq 0$ and the weight of ψ on E_{l+1} is one plus the weight of ψ on E_l.

Finally, the Morse index of f is calculated using the following result.

Proposition 3 ([GLM]). *The dimension of the eigenspace of the action of ψ on the tangent space for the eigenvalue $-k$ equals the first hypercohomology group of a complex*

$$C_k^\bullet : U_k \to \bar{U}_k \otimes K(D)$$

where

$$U_k = \oplus_{j-i=2k} \operatorname{ParHom}(E_i, E_j) \quad \bar{U}_k = \oplus_{j-i=2k+1} \operatorname{SParHom}(E_i, E_j) \tag{8}$$

Thus in the $U(2,1)$ case we have the following possibilities. If (E,Φ) is a critical point then it can be of one of these three following forms:

$$\begin{array}{lll} E = E_0 \oplus E_1, & \operatorname{rk}(E_0) = 1, & \operatorname{rk}(E_1) = 2 \\ E = E_0 \oplus E_1, & \operatorname{rk}(E_0) = 2, & \operatorname{rk}(E_1) = 1 \\ E = E_0 \oplus E_1 \oplus E_2 & \operatorname{rk}(E_i) = 1, & i = 1,2,3. \end{array}$$

These form, critical subvarieties of types $(\operatorname{rk}(E_0), \operatorname{rk}(E_1)$ or $(\operatorname{rk}(E_0), \operatorname{rk}(E_1), \operatorname{rk}(E_2))$ particularly in this case $(1,2)$, $(2,1)$, and $(1,1,1)$ respectively. And this critical subvarieties can be identified with triples of type $(1,2,d_1,d_0;\alpha_1,\alpha_2,\eta)$, $(2,1,d_1,d_0;\eta,\alpha_1,\alpha_2)$ and chains of type $(1,1,1,d_2,d_1,d_0;\alpha_{\varpi(1)},\eta,\alpha_{\varpi(2)})$. Where by *type* of a triple or a chain we mean, a system of numbers that give some topological invariants of this objects, they are the ranks, degrees and parabolic systems of weights of each parabolic bundle conforming the triples or the chain respectively.

Observe that the critical varieties of type $(1,2)$ and type $(2,1)$ consist of parabolic Higgs bundles for which either $\gamma = 0$ or $\beta = 0$ respectively. From (6) and using the definition of τ we get that they are minima for the Morse function f and, as proved in [GLM], they are the only ones. Hence its Morse index is zero.

In the cases $(2,1)$ and $(1,2)$ where $E = E_0 \oplus E_1$ the critical submanifold will be identified with certain moduli spaces of parabolic triples. However in the third case, where $E = E_0 \oplus E_1 \oplus E_2$, we will be dealing with parabolic chains. This is the reason for restricting attention to $p = 2$ and $q = 1$. If we would like to compute the Betti numbers for higher values of p and q we will have to deal with more general parabolic chains that the ones appearing here, and this tool has not been developed yet. This is left to future work.

In the following sections we will calculate the Poincaré polynomials which take part in the formula in (7), that is for the moduli space $\mathcal{U}$ of parabolic $U(2,1)$-parabolic Higgs bundles

$$P_t(\mathcal{U}) = \begin{cases} P_t\mathcal{N}_{(2,1)} + P_t\mathcal{N}_{(1,1,1)} & \text{for } \tau > 0 \\ P_t\mathcal{N}_{(1,2)} + P_t\mathcal{N}_{(1,1,1)} & \text{for } \tau < 0 \end{cases} \tag{9}$$

where we denote $P_t\mathcal{N}_{(1,2)}$ the contribution on the Poincaré polynomial of $\mathcal{U}$ of the subvariety of type $(1,2)$, $P_t\mathcal{N}_{(2,1)}$ is the contribution of the subvariety of type $(2,1)$ and $P_t\mathcal{N}_{(1,1,1)}$ is the contributions from all critical subvarieties of type $(1,1,1)$. Through these sections our computations will depend on some variables that we have mentioned above: the Toledo invariant τ of the moduli space, and the degree $\Delta = a+b$ of E. Recall that by Proposition 1 we can suppose $\tau < 0$ and $\Delta \equiv 0(3)$.

It is known that for fixed rank, and for different choices of degrees and generic weights the moduli spaces of parabolic Higgs bundles $\mathcal{M}$ have the same Poincaré polynomial (see

[GGM]), so it is possible to choose the weights conveniently for such calculation for $\mathcal{M}$. We have seen that $\mathcal{U} \subset \mathcal{M}$ is a subvariety, and our calculation of its Poincaré polynomial will show that the same phenomenon does not happen for $\mathcal{U}$. The Poincaré polynomial of $\mathcal{U}$ depends on the generic weights. We shall see this very explicitly in our calculations for one marked point.

3. Contribution to Poincaré polinomial from critical subvarieties of type (1,1,1).

We start with the case where $E = V \oplus W$ splits in three line bundles $E = E_0 \oplus E_1 \oplus E_2$ where E_0 and E_2 are contained in V, together with strongly parabolic homomorphisms $\Phi_0 = \gamma|_{E_0} : E_0 \to E_1 \otimes K(D)$ and $\Phi_1 = \beta|_{E_1} : E_1 \to E_2 \otimes K(D)$.

We denote along this section $d_i = \deg(E_i)$ so $\Delta = d_0 + d_1 + d_2 = d_0 + b + d_2$ i.e. $a = d_0 + d_2$ and $b = d_1$.

The distributions of the weights for E_0 and E_2 are given by a set of injective maps $\varpi = \{\varpi_x : \{1,2\} \to \{1,2\};\ x \in D\}$ such that the weight of E_0 at $x \in D$ is $\alpha_{\varpi(1)_x}(x)$ and the weight of E_2 at $x \in D$ is $\alpha_{\varpi(2)_x}(x)$.

Proposition 4. *The Morse index for the critical submanifolds of type (1,1,1) depends on d_0 and ϖ, and it is given by*

$$\lambda_{\mathcal{N}_{(1,1,1)}}(d_0, \varpi) = 2g - 2 + 2(2d_0 - \Delta + b) + 2(s - v)) \tag{10}$$

where $v = \sharp\{x \in D;\ \alpha_{\varpi_x(1)}(x) > \alpha_{\varpi_x(2)}(x)\}$, and s is the number of marked points.

Proof. By proposition 3 the Morse index equals the dimension of $\mathbb{H}^1(C_1^\bullet)$ where $C_1^\bullet$ is the complex

$$\mathrm{ParHom}\,(E_0, E_2) \to 0.$$

Using the long exact sequence for this complex we get $\mathbb{H}^0(C_1^\bullet) = 0$ since it is isomorphic to $H^0(\mathrm{ParHom}\,(E_0, E_2))$ and, the last is equal to zero since its degree is less than zero. Hence,

$$\begin{aligned}
\frac{1}{2}\lambda_{\mathcal{N}_{(1,1,1)}} &= \dim T_E \mathcal{U}_{<0} = \dim \mathbb{H}^1(C_1^\bullet) \\
&= \dim H^1(\mathrm{ParHom}\,(E_0, E_2)) = -\chi(\mathrm{ParHom}\,(E_0, E_2)) \\
&= -\deg(\mathrm{ParHom}\,(E_0, E_2)) - \mathrm{rk}\,(\mathrm{ParHom}\,(E_0, E_2))(1 - g) \\
&= d_0 - d_2 + s - \sum_{x \in D} \dim \mathrm{ParHom}\,(E_0, E_2)_x + g - 1.
\end{aligned}$$

Hence, $\lambda_{\mathcal{N}_{(1,1,1)}} = 2g - 2 + 2(2d_0 + b - \Delta) + 2(s - v)$, where $v = \sharp\{x \in D; \alpha_{\varpi_x(1)}(x) \leq \alpha_{\varpi_x(2)}(x)\}$. □

Remark 5. The Proposition above proves also that $\lambda_{\mathcal{N}(1,1,1)}$ depend only on d_0 and ϖ, the data that give us how splits V into E_0 and E_2. So we may decompose $\mathcal{N}(1,1,1) = \bigcup_{d_0, \varpi} \mathcal{N}(d_0, \varpi)$.

From now on we denote

$$v_1 = \sharp\{x \in D;\, \alpha_{\varpi(1)}(x) < \eta(x)\}$$
$$v_2 = \sharp\{x \in D;\, \eta(x) < \alpha_{\varpi(2)}(x)\}.$$

Proposition 6. *Assume $\tau < 0$, let $\mathcal{N}_{(1,1,1)}$ be the union of critical submanifolds of type $(1,1,1)$ parametrized by d_0 and ϖ, i.e. $\mathcal{N}_{(1,1,1)} = \bigcup_{d_0,\varpi} \mathcal{N}(d_0,\varpi)$. The map*

$$\begin{array}{rcl} \mathcal{N}(d_0,\varpi) & \to & \mathrm{Jac}^{d_0} X \times S^{m_1}X \times S^{m_2}X \\ (E_0 \oplus E_1 \oplus E_2, \Phi_0, \Phi_1) & \mapsto & (E_0, \mathrm{div}\,(\Phi_0), \mathrm{div}\,(\Phi_1)) \end{array}$$

where

$$\begin{array}{rcl} m_1 & = & \deg(\mathrm{SParHom}\,(E_0,E_1) \otimes K(D)) = b - d_0 + 2g - 2 + v_1 \\ m_2 & = & \deg(\mathrm{SParHom}\,(E_1,E_2) \otimes K(D)) = \Delta - d_0 - 2b + 2g - 2 + v_2 \end{array}$$

is an isomorphism, in particular there is only one component for fixed d_0 and ϖ. Furthermore, d_0 the degree of E_0 is lower bounded by $\bar{d}_0$, that is,

$$d_0 \geq \bar{d}_0 = \left[\frac{1}{3}\left(\Delta + \sum_{x\in D}(\eta(x) + \alpha_{\varpi_x(2)}(x) - 2\alpha_{\varpi_x(1)}(x))\right) + 1\right] \tag{11}$$

where $[k]$ denote the entire part of k.

Proof. The isomorphism is obvious (see [GGM]). The stability condition on E applied on the subbundles E_2 and $E_1 \oplus E_2$, together with the formula $d_2 = \Delta - b - d_0$ gives the following two bounds for d_0:

$$2\Delta - 3b - \sum_{x\in D}(\alpha_{\varpi_x(1)}(x) + \eta(x) - 2\alpha_{\varpi_x(2)}(x)) < 3d_0 \tag{12}$$

$$\Delta - \sum_{x\in D}(2\alpha_{\varpi_x(1)}(x) - \eta(x) - \alpha_{\varpi_x(2)}(x)) < 3d_0. \tag{13}$$

To determine which is the appropriate bound we subtract these two inequalities. This subtraction gives a multiple of τ , hence $\bar{d}_0$ depends on wether τ is negative or positive. □

Remark 7. The condition on the weights being generic implies that τ can not be zero. This is because $\tau = 0$ implies that $2\eta(x) - \alpha_1(x) - \alpha_2(x) = \Delta - 3b$, and if that happens then there is a $U(2,1)$-parabolic Higgs subbundle $(V, \Phi = 0)$ non-stable but semistable.

Remark 8. Note that the values m_1 and m_2 depend on ϖ_x and $\bar{d}_0$.

Remark 9. We chose $\tau < 0$ for computability reasons.

Denote $\varpi = \{\varpi_x\}_{x\in D}$.

Theorem 10. *The Poincaré polynomial of the critical submanifold $\mathcal{N}(d_0,\varpi)$ is*

$$P_t(\mathcal{N}(d_0,\varpi)) = (1+t)^{2g}\,\mathrm{Coeff}_{\,x^0y^0}\left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)x^{m_1}} \cdot \frac{(1+yt)^{2g}}{(1-y)(1-yt^2)y^{m_2}}\right)$$

where m_1 and m_2 are the same as in Proposition 6.

Proof. Use Macdonald's formula for the Poincaré polynomial of the symetric product (see [M]). □

Now, in order to get the contribution of all the subvarieties of type $(1,1,1)$ in $P_t(\mathcal{U})$ we have to sum over all $d_0 \geq \bar{d}_0$ and all possibilities of ϖ.

$$\begin{aligned}
P_t(\mathcal{N}_{(1,1,1)}) &= \sum_{d_0,\varpi} t^{\lambda_{\mathcal{N}(d_0,\varpi)}} P_t(\mathcal{N}(d_0,\varpi)) \\
&= \sum_{d_0,\varpi} \left(t^{2g-2+2(b-\Delta)+4d_0+2(s-v)} \operatorname{Coeff}_{x^0y^0} \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)x^{m_1}} \cdot \frac{(1+yt)^{2g}}{(1-y)(1-yt^2)y^{m_2}} \right) \right) \\
&= \operatorname{Coeff}_{x^0y^0} \left(\sum_{\varpi} \frac{t^{2g-2+2b-2\Delta+2s}(1+xt)^{2g}(1+yt)^{2g}}{(1-x)(1-xt^2)x^{b+2g-2}(1-y)(1-yt^2)y^{\Delta-2b+2g-2}} \cdot \frac{t^{4\bar{d}_0}x^{\bar{d}_0}y^{\bar{d}_0}}{t^{2v}x^{v1}y^{v2}} \right) \\
&= \operatorname{Coeff}_{x^0y^0} \left(\frac{t^{2g-2+2b-2\Delta+2s}(1+xt)^{2g}(1+yt)^{2g}}{(1-x)(1-xt^2)x^{b+2g-2}(1-y)(1-yt^2)y^{\Delta-2b+2g-2}} \cdot \sum_{\varpi} \frac{t^{4\bar{d}_0}x^{\bar{d}_0}y^{\bar{d}_0}}{t^{2v}x^{v1}y^{v2}} \right)
\end{aligned}$$

Thus, we have to compute the following sum

$$\sum_{\varpi_x} \frac{t^{4\bar{d}_0}x^{\bar{d}_0}y^{\bar{d}_0}}{t^{2v}x^{v_1}y^{v_2}}. \tag{14}$$

The variables depend also on the weights $\alpha_1(x)$, $\alpha_2(x)$ and $\eta(x)$, and the distribution functions ϖ_x.

4. Computations for one puncture for $\mathcal{N}_{(1,1,1)}$.

From now on we consider the case of one puncture to get more explicit formulas, so we denote $\alpha_i = \alpha_i(x)$ for $i = 1, 2$ and $\eta = \eta(x)$. We abbreviate ϖ_x to ϖ.

We have to consider the following cases for the possible distributions of the weights,

Table 1. Weight distributions.

S_1	$\eta < \alpha_1 < \alpha_2$
$S_1(a)$	$\alpha_2 - \alpha_1 > \alpha_1 - \eta$
$S_1(b)$	$\alpha_2 - \alpha_1 < \alpha_1 - \eta$
S_3	$\alpha_1 < \alpha_2 < \eta$
$S_3(a)$	$\alpha_2 - \eta > \alpha_2 - \alpha_1$
$S_3(b)$	$\alpha_2 - \eta < \alpha_2 - \alpha_1$

S_2	$\alpha_1 < \eta < \alpha_2$
S_4	$\eta = \alpha_1 < \alpha_2$
S_5	$\alpha_1 < \alpha_2 = \eta$
S_6	$\alpha_1 = \alpha_2 = \eta$
S_7	$\eta < \alpha_1 = \alpha_2$
S_8	$\alpha_1 = \alpha_2 < \eta$

Theorem 11. *The contributions to the Poincaré polynomial of the union of the subvarieties of type* $(1,1,1)$ *when* $\tau < 0$ *and one marked point are classified by the possibilities for the distribution of the weights of* E *shown in Table 1. They are the following,*

(i) *For* $S_1(a)$:

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2\,b-\frac{2\,\Delta}{3}+2\,g}\,x^{2-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,y^{1+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}\,(1+t^2\,x\,y)}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)}\right).$$

(ii) *For* $S_1(b)$:

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{-2+2\,b-\frac{2\,\Delta}{3}+2\,g}\,(1+t^2)\,x^{2-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,y^{1+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)}\right).$$

(iii) *For* S_2 :

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2\,b-\frac{2\,\Delta}{3}+2\,g}\,(1+t^2)\,x^{2-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,y^{2+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)}\right).$$

(iv) *For* $S_3(a)$:

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2+2\,b-\frac{2\,\Delta}{3}+2\,g}\,(1+t^2)\,x^{2-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,y^{3+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)}\right).$$

(v) *For* $S_3(b)$:

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2\,b-\frac{2\,\Delta}{3}+2\,g}\,x^{1-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,y^{2+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}\,(1+t^2\,x\,y)}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)}\right).$$

(vi) *For* S_4 :

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2\,b-\frac{2\,\Delta}{3}+2\,g}\,x^{2-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,(1+t^2\,x)\,y^{2+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)}\right).$$

(vii) *For* S_5 :

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2\,b-\frac{2\,\Delta}{3}+2\,g}\,x^{2-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,y^{2+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}\,(1+t^2\,y)}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)}\right).$$

(viii) *For* S_6 :

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{2\,t^{2+2\,b-\frac{2\,\Delta}{3}+2\,g}\,x^{3-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,y^{3+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)}\right).$$

(ix) *For* S_7 :

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{2\,t^{-2+2\,b-\frac{2\,\Delta}{3}+2\,g}\,x^{2-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,y^{1+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)}\right).$$

(x) *For* S_8 :

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{2\,t^{-2+2\,b+4\left(1+\frac{\Delta}{3}\right)-2\,\Delta+2\,g}\,x^{2-b+\frac{\Delta}{3}-2\,g}\,(1+t\,x)^{2\,g}\,y^{3+2\,b-\frac{2\,\Delta}{3}-2\,g}\,(1+t\,y)^{2\,g}}{(1-x)\,(1-t^2\,x)\,(1-y)\,(1-t^2\,y)}\right).$$

Proof. Compute the values of $\bar{d}_0$, v_1. v_2 and v for each possible distribution of the weights, then we obtain the values for the sum in (14) for each case S_i.

The value for $\bar{d}_0$ depends on the distribution of the weights and is, in the case $\varpi = \mathrm{Id}$, $\bar{d}_0 = \frac{\Delta}{3} + 1$ for all S_i except for $S_1(b)$ and S_7 where $\bar{d}_0 = \frac{\Delta}{3}$. When $\varpi \neq \mathrm{Id}$, $\bar{d}_0 = \frac{\Delta}{3}$ for all S_i except for $S_3(a)$, S_6 and S_8 where it is $\bar{d}_0 = \frac{\Delta}{3} + 1$.

□

5. Poincaré Polynomial for critical subvarieties of type $(1, 2)$.

Following our previous discussion the critical subvarieties of type $(1, 2)$ and $(2, 1)$ can be identified with the moduli of $(2g - 2)$-stable parabolic triples of type $(2, 1, a + 4g - 4, b; \alpha_1, \alpha_2, \eta)$ and $(1, 2, b + 2g - 2, a; \alpha_1, \alpha_2, \eta)$ respectively. So we recall the basics of parabolic triples from [GGM].

¿From Proposition 1 we restrict to the case when $\tau > 0$, note that by definition the Morse function f forces $\gamma = 0$ when $\tau > 0$. Hence, for our analysis we only have to consider the critical subvarieties of type $(1, 2)$, that is $(2g - 2)$-stable parabolic triples of type $(2, 1, a + 4g - 4, b; \alpha_1, \alpha_2, \eta)$.

A *parabolic triple* is a holomorphic triple $T = (T_1, T_2, \phi)$ where T_1 and T_2 are parabolic bundles over X, and $\phi : T_2 \to T_1(D)$ is a strongly parabolic homomorphism, i.e. an element $\phi \in H^0(\mathrm{SParHom}\,(T_2, T_1(D)))$. We call *parabolic system of weights* for the triple (T, ϕ) to the vector $\alpha = (\alpha^1, \alpha^2)$ where α^i is the system of weights of T_i with $i = 1, 2$. The *type* of a parabolic triple is a n-tuple $(r_1, r_2, d_1, d_2; \alpha_1(x), \dots, \alpha_{r(x)}(x), \eta_1(x), \dots, \eta_{r'(x)}(x)$, where $r_i = \mathrm{rk}\,(T_i)$, $d_i = \deg(T_i)$, α is the parabolic system of weights of T_1 and η is the parabolic system of weights of T_2.

A parabolic triple $T' = (T'_1, T'_2, \phi')$ is a *parabolic subtriple* of $T = (T_1, T_2, \phi)$ if $T'_i \subset T_i$ are parabolic subbundles for $i = 1, 2$ and $\phi'(T'_2) \subset T'_1(D)$ where ϕ' is the restriction of ϕ to T'_2.

For any $\sigma \in \mathbb{R}$ the *σ-parabolic degree* of T is defined to be

$$\mathrm{pardeg}_{\,\sigma}(T) = \mathrm{pardeg}\,(T_1) + \mathrm{pardeg}\,(T_2) + \sigma\,\mathrm{rk}\,(T_2).$$

In the following we denote $r_1 = \mathrm{rk}\,(T_1)$ and $r_2 = \mathrm{rk}\,(T_2)$. Thus we have a notion of stability for a fixed parameter. Let σ be a real number. We define the *σ-slope of a triple* (T_1, T_2, ϕ) as

$$\mathrm{par}\mu_{\,\sigma}(T) = \frac{\mathrm{pardeg}\ T_1 + \mathrm{pardeg}\ T_2}{r_1 + r_2} + \sigma\frac{r_2}{r_1 + r_2}. \tag{15}$$

T is called σ-stable (resp. σ-semistable) if for any non-zero proper subtriple T' we have $\mathrm{par}\mu_{\,\sigma}(T') < \mathrm{par}\mu_{\,\sigma}(T)$ (resp.$\leq$).

Proposition 12. *Subvarieties of type* $(1, 2)$ *and type* $(2, 1)$ *correspond with* σ*-stable triples for* $\sigma = 2g - 2$.

Proof. In the case of the study of critical varieties of type $(1,2)$, Simpson's theorem says that we have a variation of the Hodge structure like this:

$$E = E_0 \oplus E_1, \quad \Phi = \gamma : E_0 = W \to E_1 \otimes K(D) = V \otimes K(D),$$

with $\operatorname{rk}(E_0) = 1$ and $\operatorname{rk}(E_1) = 2$. Therefore we get $T = (E_1 \otimes K, E_0, \beta)$ a parabolic triple of type $(2, 1, a + 4g - 4, b; \alpha_1, \alpha_2, \eta)$.

Analogously in the case of the study of critical varieties of type $(2,1)$ Simpson's theorem give us a variation of the Hodge structure like before.

$$E_0 \oplus E_1 \quad \Phi = \beta : E_0 = V \to E_1 \otimes K(D) = W \otimes K(D)$$

With $\operatorname{rk}(E_0) = 2$ and $\operatorname{rk}(E_1) = 1$.

Hence in the case of critical varieties of type $(2,1)$ we have to study parabolic triples of type $(1, 2, b + 2g - 2, a; \eta, \alpha_1, \alpha_2)$ with $T = (T_1, T_2, \phi) = (E_1 \otimes K, E_0, \beta)$. □

Proposition 13. *The Morse index for critical submanifolds of type* $(2,1)$ *and type* $(1,2)$ *is* $\lambda_{\mathcal{N}} = 0$. *In particular it does not depend on the weights.*

Proof. This is clear since these subvarieties are minima for the Morse function. □

6. Computations for one puncture for $\mathcal{N}_{(1,2)}$.

Let ϖ be a fixed distribution of the weights over the marked point x. In the following v_1, v_2 and v_3 are given by

$$\begin{aligned} v_1 &= \begin{cases} 1 & \text{if} \quad \eta < \alpha_{\varpi(2)} \\ 0 & \text{otherwise} \end{cases} \\ v_2 &= \begin{cases} 1 & \text{if} \quad \eta < \alpha_{\varpi(1)} \\ 0 & \text{otherwise} \end{cases} \\ v_3 &= \begin{cases} 1 & \text{if} \quad \alpha_{\varpi(1)} < \alpha_{\varpi(2)} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Let $\sigma > \sigma_m$ be a non-critical value. For any ϖ, $\bar{d}_M$ we define,

$$\bar{d}_M = \left[\frac{1}{3} \left(\Delta + \alpha_{\varpi(2)} + \eta - 2\alpha_{\varpi(1)} + \sigma \right) + 1 \right].$$

Proposition 14. *The Poincaré polynomial of the moduli of parabolic triples* $T = (T_1, T_2, \phi)$ *of type* $(2, 1, \bar{d}_1, \bar{d}_2; \alpha, \eta)$ *and one marked point is*

(16)

$$\operatorname{Coeff}_{x^0} \frac{(1+t)^{4g}(1+xt)^{2g}}{(1-t^2)(1-x)(1-xt^2)} \sum_{\varpi} x^{\bar{d}_M - \bar{d}_1 + \bar{d}_2 - v_1} \left(\frac{t^{2\bar{d}_1 - 2\bar{d}_2 + 2v_2 + 2v_3 - 2\bar{d}_M}}{1 - t^{-2}x} - \frac{t^{-2\bar{d}_1 + 2g - 2v_3 + 4\bar{d}_M}}{1 - t^4 x} \right)$$

Proof. Rewrite theorem 6.5 from [GGM] for this concrete conditions. □

Remark 15. In [GGM] the Poincaré polynomial is computed under the assumption of generic distinct weights but this formula does not use the assumption.

Theorem 16. *The Poincaré polynomials for the critical variety of type* $(1,2)$ *when* $\tau > 0$ *for one marked point are classified by the possibilities for the distribution of the weights of* E *given in Table 1 and, they are the following,*

(i) *For* $S_1(a)$ *and* S_7

$$\mathrm{Coeff}_{x^0}\left(\frac{(1+t)^{4g}\,x^{1+2b-\frac{2}{3}\Delta-2g}(1+t\,x)^{2g}}{t^{4b+\frac{2}{3}\Delta-2g}(-1+t^2)\,(t^2-x)\,(-1+x)\,(-1+t^2\,x)\,(-1+t^4\,x)}\right.$$
$$\left.\left(t^{6b}\,x-t^{4+6b}\,x+t^{2\Delta+2g}\,(1+x)-t^{4+2\Delta+2g}\,x\,(1+x)+t^{2+6b}\,(-1+x^2)\right)\right).$$

(i) *For* $S_1(b)$

$$\mathrm{Coeff}_{x^0}\left(-\frac{(1+t)^{4g}\,\left(1+t^2\right)\,x^{1+2b-\frac{2}{3}\Delta-2g}\,(1+t\,x)^{2g}\,\left(t^{2+6b}-t^{2+2\Delta+2g}-t^{6b}\,x+t^{6+2\Delta+2g}\,x\right)}{t^{2+4b+\frac{2}{3}\Delta-2g}\,\left(-1+t^2\right)\,\left(t^2-x\right)\,\left(-1+x\right)\,\left(-1+t^2\,x\right)\,\left(-1+t^4\,x\right)}\right).$$

(ii) *For* S_2 *and* S_4

$$\mathrm{Coeff}_{x^0}\left(-\frac{(1+t)^{4g}\,\left(1+t^2\right)\,x^{2+2b-\frac{2}{3}\Delta-2g}\,(1+t\,x)^{2g}\,\left(t^{4+6b}-t^{2\Delta+2g}-t^{2+6b}\,x+t^{4+2\Delta+2g}\,x\right)}{t^{2+4b+\frac{2}{3}\Delta-2g}\,\left(-1+t^2\right)\,\left(t^2-x\right)\,\left(-1+x\right)\,\left(-1+t^2\,x\right)\,\left(-1+t^4\,x\right)}\right).$$

(iii) *For* $S_3(a)$*,* S_6 *and* S_8

$$\mathrm{Coeff}_{x^0}\left(-\frac{(1+t)^{4g}\,\left(1+t^2\right)\,x^{3+2b-\frac{2}{3}\Delta-2g}\,(1+t\,x)^{2g}\,\left(t^{8+6b}-t^{2\Delta+2g}-t^{6+6b}\,x+t^{4+2\Delta+2g}\,x\right)}{t^{4+4b+\frac{2}{3}\Delta-2g}\,\left(-1+t^2\right)\,\left(t^2-x\right)\,\left(-1+x\right)\,\left(-1+t^2\,x\right)\,\left(-1+t^4\,x\right)}\right).$$

(iv) *For* $S_3(b)$ *and* S_5

$$\mathrm{Coeff}_{x^0}\left(\frac{(1+t)^{4g}\,x^{2+2b-\frac{2\Delta}{3}-2g}\,(1+t\,x)^{2g}}{t^{2+4b+\frac{2}{3}\Delta-2g}\,(-1+t^2)\,(t^2-x)\,(-1+x)\,(-1+t^2\,x)\,(-1+t^4\,x)}\right.$$
$$\left.\left(t^{2+6b}\,x-t^{6+6b}\,x+t^{2\Delta+2g}\,(1+x)-t^{4+2\Delta+2g}\,x\,(1+x)+t^{4+6b}\,(-1+x^2)\right)\right).$$

Proof. We only have to apply Proposition 14 using the different values for v_1 and v_2 on each case. Use also that $v_3 = 1$ if $\varpi = \mathrm{Id}$ and equal to zero otherwise. Hence we have different values for $\bar{d}_M$ depending on the distribution of the weights. These are, when $\varpi = \mathrm{Id}$, $\bar{d_M} = \frac{\Delta}{3}+2g-1$ fora all S_i except for $S_1(a)$ and S_7 where $\bar{d_M} = \frac{\Delta}{3}+2g-2$. And when $\varpi \neq \mathrm{Id}$, $\bar{d_M} = \frac{\Delta}{3}+2g-2$ for all S_i except for S_6 and S_8 where $\bar{d_M} = \frac{\Delta}{3}+2g-1$. □

7. Poincaré polinomial of $\mathcal{U}$ with one marked point

Summarizing, we are using Morse-Bott theory in order to calculate the Poincaré polynomial of $\mathcal{U}$ the moduli of stable $U(2,1)$ parabolic Higgs bundles with degrees $a = \deg(V)$ and $b = \deg(W)$. Therefore we have described the critical subvarieties of $\mathcal{U}$ for the Morse function f. These consist of several subvarieties of type $(1,1,1)$ parametrized by (d_0, ϖ) and, depending on τ, and one subvariety corresponding to the minima of f, which is of type $(1,2)$ when $\tau < 0$ and of type $(2,1)$ when $\tau > 0$.

Corollary 17. *The Poincaré polinomial of* $U(2,1)$ *parabolic Higgs bundles for* $\tau < 0$ *is given by*

$$P_t(\mathcal{U}) = P_t(\mathcal{N}_{(1,1,1)}) + P_t(\mathcal{N}_{2g-2}(2,1,a+4g-4,b))$$

Corollary 18. *The Poincaré polynomial of $U(2,1)$ parabolic Higgs bundles for $\tau > 0$ is given by*
$$P_t(\mathcal{M}(a,b)) = P_t(\mathcal{N}_{(1,1,1)}) + P_t(\mathcal{N}_{2g-2}(1,2,b+2g-2,a))$$

We can compute the Poincaré polynomial of the moduli space of parabolic $U(2,1)$-Higgs bundles for specific values of a, b, and g using a computer algebra system.

Note that in order to give an example we fix a, b (so we fix Δ) such that Δ is equal to zero modulo 3 and $\tau < 0$. The other varieties, with different values for Δ and tau are diffeomorphic to these from Proposition 1

Fix $g = 1$, degrees $a = b = 0$, and $\alpha_1 < \alpha_2 < \eta$ such that $\eta - \alpha_2 < \alpha_2 - \alpha_1$, that is, we are in case $S_3(b)$ of Table 1.

The contribution from the critical subvarieties of type $(1,1,1)$ is
$$P_t(\mathcal{N}_{(1,1,1)}) = t^2 + 2\,t^3 + t^4,$$
and from the critical subvariety of type $(1,2)$ is
$$P_t(\mathcal{N}_{2g-2}(2,1,a+4g-4,b)) = 1 + 4\,t + 6\,t^2 + 4\,t^3 + t^4.$$

Hence, the Poincaré polynomial for $\mathcal{U}(0,0)$ with $g = 1$, when $\alpha_1 < \alpha_2 < \eta$ such that $\eta - \alpha_2 < \alpha_2 - \alpha_1$, is
$$P_t(\mathcal{U}) = 1 + 4\,t + 7\,t^2 + 6\,t^3 + 2\,t^4.$$

As an example of the phenomena we have talked above, different polynomials for different weights we give another example for same genus and degrees but for case S_5.

The contribution from the critical subvarieties of type $(1,1,1)$ is
$$P_t(\mathcal{N}_{(1,1,1)}) = t^2,$$
and from the critical subvariety of type $(1,2)$ is again
$$P_t(\mathcal{N}_{2g-2}(2,1,a+4g-4,b)) = 1 + 4\,t + 6\,t^2 + 4\,t^3 + t^4.$$

Hence, the Poincaré polynomial for $\mathcal{U}(0,0)$, when $g = 1$, one marked point and, $\alpha_1 < \alpha_2 = \eta$, is
$$P_t(\mathcal{U}) = 1 + 4\,t + 7\,t^2 + 4\,t^3 + t^4$$

Note that can only choose degrees $a = b = 0$ for distributions of weights S_3 , S_5 and S_8

Proposition 19. *The complex dimension of the moduli space of parabolic $U(2,1)$-Higgs bundles is $1 + 9(g-1) + \sum_{x \in D}(3 - c)$ where c is the number of weights $\alpha_i(x)$ equal to $\eta(x)$.*

Proof. Rewrite Proposition 3.4 from [GLM]. □

Hence, in the examples above the real dimension of $\mathcal{U}$ is 6 and does not coincides with the degree of the polynomials. Also it is interesting the fact that one of the polynomials satisfies Poincaré duality and the other does not.

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